Solutions Resit Exam — Partial Differential Equations

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Duration: 3 hours

Question 1 (15 points)

Consider the equation

$$yu_x - u_y = 0, \tag{1}$$

where $u = u(x, y)$.

a. (7 pt) Find the general solution of Eq. (1).

b. (3 pt) Find the solution of Eq. (1) with the auxiliary condition $u(x, 0) = 2\sin x$.

Consider now the equation

$$yu_x - u_y = u^2. \tag{2}$$

c. (5 pt) Find the general solution of Eq. (2) without any auxiliary conditions, using the substitution $u(x, y) = \frac{1}{y - w(x,y)}$.

Solution

a. We solve the equation for the characteristic curves

$$\frac{dy}{dx} = \frac{-1}{y},$$

which gives

$$\frac{y^2}{2} = -x + C_1,$$

where C_1 is the constant of integration. Solving for $C = 2C_1$ we get

$$C = y^2 + 2x.$$

Since $y^2 + 2x$ is constant along the characteristic curves we conclude that the solution of the problem has the general form

$$u(x, y) = f(y^2 + 2x),$$

where f is an arbitrary function of one variable.

b. Applying the general solution we find

$$u(x, 0) = f(2x) = 2\sin x.$$

Therefore $f(s) = 2\sin(s/2)$, and the solution we are after is

$$u(x, y) = 2\sin\Big(\frac{y^2 + 2x}{2}\Big).$$

c. We have

$$u_x = \frac{1}{(y-w)^2} w_x, \quad u_y = \frac{1}{(y-w)^2}(-1 + w_y).$$

Then Eq. (2) gives

$$\frac{1}{(y-w)^2}(yw_x + 1 - w_y) = \frac{1}{(y-w)^2},$$

which can be simplified, to

$$yw_x - w_y = 0,$$

which is exactly Eq. (1), and for which we know that the general solution is

$$w = f(y^2 + 2x).$$

Therefore, the general solution for Eq. (2) is

$$u = \frac{1}{y - f(y^2 + 2x)},$$

with f an arbitrary function.

Question 2 (15 points)

Consider the equation

$$u_{xx} - 2u_{xy} - u_{yy} = 0, \tag{3}$$

where $x, y \in \mathbb{R}$.

a. **(3 pt)** What is the type (elliptic / hyperbolic / parabolic) of Eq. (3)? Explain your answer.

b. **(8 pt)** Find a linear transformation $(x, y) \to (s, t)$ that reduces Eq. (3) to one of the standard forms $u_{ss} + u_{tt} = 0$, $u_{ss} - u_{tt} = 0$, or $u_{ss} = 0$.

c. **(4 pt)** Find the general solution of Eq. (3).

Solution

a. We have $a_{11} = 1$, $a_{22} = -1$, and $a_{12} = -1$. Therefore

$$a_{12}^2 > a_{11}a_{22},$$

and Eq. (3) is *hyperbolic*.

b. Since the equation is hyperbolic the standard form is $u_{ss} - u_{tt} = 0$, or $(\partial_s^2 - \partial_t^2)u = 0$. Write the original equation as

$$\mathcal{L}u = 0,$$

where

$$\mathcal{L} = \partial_x^2 - 2\partial_x\partial_y - \partial_y^2.$$

Then

$$\mathcal{L} = (\partial_x - \partial_y)^2 - 2\partial_y^2,$$

so we can set $\partial_s = \partial_x - \partial_y$ and $\partial_t = \sqrt{2}\partial_y$, that is,

$$\begin{pmatrix} \partial_s \\ \partial_t \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}.$$

The corresponding coordinate transformation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix},$$

or

$$x = s, \quad y = \sqrt{2}t - s,$$

which can be inverted to give

$$s = x, \quad t = (x + y)/\sqrt{2}.$$

c. We have transformed Eq. (3) to $u_{ss} - u_{tt} = 0$ which is the wave equation (with $c = 1$). The latter has the general solution

$$u(s, t) = f(s + t) + g(s - t).$$

This means that the original equation has the general solution

$$u(x, y) = f\Big(\frac{(\sqrt{2} + 1)x + y}{\sqrt{2}}\Big) + g\Big(\frac{(\sqrt{2} - 1)x - y}{\sqrt{2}}\Big).$$

Question 3 (20 points)

Consider the Laplace equation $\Delta u = 0$ in the domain

$$R = \{(x, y) : 0 \leq x \leq \pi, 0 \leq y \leq \pi\},$$

with the boundary conditions $u(x, \pi) = \sin x + \frac{1}{3}\sin(3x)$, and $u(x, 0) = u(0, y) = u(\pi, y) = 0$.

a. (5 pt) Separate the Laplace equation in Cartesian coordinates x, y using the ansatz $u(x, y) = X(x)Y(y)$ and write two ordinary differential equations, one for X and one for Y.

b. (7 pt) Solve the eigenvalue equation for X for the given boundary conditions (find eigenvalues and eigenfunctions). Consider known that the problem has no complex eigenvalues but check for positive, negative, or zero eigenvalues.

c. (3 pt) Solve the differential equation for Y.

d. (5 pt) Write the general solution $u(x, y)$ for boundary conditions $u(x, \pi) = h(x)$, $u(x, 0) = u(0, y) = u(\pi, y) = 0$ where $h(x)$ is an arbitrary function and then give the solution for the specific boundary conditions in this problem.

Solution

a. Substituting $u(x, y) = X(x)Y(y)$ into the equation

$$\Delta u = u_{xx} + u_{yy} = 0$$

we get

$$X''Y + XY'' = 0.$$

Then

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

and separating the parts that depend on x from those that depend on y we find

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda,$$

where λ is a constant. Then we have the two equations

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0.$$

b. The given boundary conditions imply

$$X(0) = X(\pi) = 0.$$

For positive eigenvalues $\lambda = \beta^2$ we have the solutions

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

From here

$$X(0) = A = 0, \quad X(\pi) = A\cos(\beta\pi) + B\sin(\beta\pi) = 0.$$

Then

$$A = 0, \quad B\sin(\beta\pi) = 0,$$

and finally

$$\beta_n = n,\ n = 1, 2, 3, \ldots$$

For $\lambda = 0$ the solution is

$$X(x) = Ax + B$$

so

$$X(0) = B = 0, \quad X(\pi) = A\pi + B = 0$$

giving the trivial solution $A = B = 0$ which is rejected.
For $\lambda = -\gamma^2 < 0$ we have

$$X(x) = Ae^{\gamma x} + Be^{-\gamma x}.$$

Then

$$X(0) = A + B = 0, \quad X(\pi) = Ae^{\gamma\pi} + Be^{-\gamma\pi} = 0,$$

so

$$B = -A, \quad Ae^{-\gamma\pi}(e^{2\gamma\pi} - 1) = 0.$$

The last equation implies that either $\gamma = 0$ (so $\lambda = 0$ but we assumed $\lambda < 0$) or $A = B = 0$ giving the trivial solution so we should also reject the case of negative eigenvalues.
Finally, the eigenvalues are

$$\lambda_n = \beta_n^2 = n^2,$$

and the eigenfunctions

$$X_n(x) = \sin(nx).$$

c. The differential equation for Y is $Y'' - n^2 Y = 0$, with $n = 1, 2, 3, \ldots$. Therefore

$$Y_n(y) = A_n e^{ny} + B_n e^{-ny}.$$

d. The general solution is

$$u(x, y) = \sum_{n=1}^{\infty} X_n(x) Y_n(y) = \sum_{n=1}^{\infty} (A_n e^{ny} + B_n e^{-ny}) \sin(nx).$$

For $y = 0$ we have

$$u(x, 0) = \sum_{n=1}^{\infty} (A_n + B_n) \sin(nx) = 0$$

so

$$A_n + B_n = 0,$$

since these are the coefficients of the Fourier sine series for 0. For $y = \pi$ we have

$$u(x, \pi) = \sum_{n=1}^{\infty} A_n (e^{n\pi} - e^{-n\pi}) \sin(nx) = h(x)$$

so

$$A_n = \frac{2}{\pi(e^{n\pi} - e^{-n\pi})} \int_0^\pi h(x) \sin(nx)\, dx.$$

In the specific case here we have

$$h(x) = \sin x + \frac{1}{3}\sin(3x) = \sum_{n=1}^{\infty} A_n(e^{n\pi} - e^{-n\pi}) \sin(nx),$$

and comparing the two expressions we see that $A_1 = [e^\pi - e^{-\pi}]^{-1}$, $A_3 = [3(e^{3\pi} - e^{-3\pi})]^{-1}$, $A_n = 0$ in all other cases. Therefore the solution for the specific boundary conditions is

$$u(x, y) = \frac{e^y - e^{-y}}{e^\pi - e^{-\pi}} \sin(x) + \frac{1}{3}\frac{e^{3y} - e^{-3y}}{e^{3\pi} - e^{-3\pi}} \sin(3x).$$

Question 4 (10 points)

Suppose that u is a harmonic function in the closed disk $D = \{r \leq 1\}$ and that

$$u = 3\sin(2\theta) - 1 \text{ for } r = 1.$$

a. (5 pt) What are the maximum and minimum values of u in D?

b. (5 pt) Find the value of u at the origin.

Solution

a. Since u is harmonic it attains its maximum and minimum values at the boundary. At the boundary we have $u = 3\sin(2\theta) - 1$. Since $-1 \leq \sin(2\theta) \leq 1$ and the values ± 1 are attained for $\theta \in [0, 2\pi]$ we conclude that $-4 \leq u \leq 2$ at the boundary and u attains the maximum value 2 and the minimum value -4 at some point on the boundary. Therefore these are also the respective maximum and minimum values on D.

b. Poisson's formula is

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\, d\phi.$$

Applying Poisson's formula for $r = 0$, $a = 1$, $h(\phi) = 3\sin(2\phi) - 1$ gives

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} (3\sin(2\phi) - 1)\, d\phi = -1.$$

Question 5 (15 points)

Consider the function

$$f(x) = x, \quad \text{with} \quad x \in [0, \pi],$$

and its Fourier sine series.

a. (2 pt) Does the Fourier sine series converge in the L^2 sense? Explain your answer.

b. (3 pt) What is the pointwise limit of the Fourier sine series for arbitrary $x \in [-\pi, \pi]$?

c. (3 pt) How does the Gibbs phenomenon manifest itself in the Fourier sine series? That is, at which point(s) in $[0, \pi]$ the Gibbs phenomenon appears and approximately how much is the "overshoot" there?

d. (7 pt) Compute the coefficients of the Fourier sine series for $f(x)$.

Solution

a. The function f is bounded in $[0, \pi]$, therefore

$$\|f\|^2 = \int_0^\pi f(x)^2\, dx < +\infty.$$

This means that the Fourier sine series converges in the L^2 sense.

b. The pointwise limit of the Fourier sine series can be deduced from the odd-periodic extension $f_{\text{ext}}(x)$ of $f(x)$ from $[0, \pi]$ to $\mathbb{R}$. This is constructed by first considering the extension of f to an odd function defined in $[-\pi, \pi]$ and then the further periodic extension to $\mathbb{R}$. This extension $f_{\text{ext}}(x)$ is discontinuous at $x = (2k+1)\pi$, $k \in \mathbb{Z}$ and $f_{\text{ext}}((2k+1)\pi^+) = -\pi$ while $f_{\text{ext}}((2k+1)\pi^-) = \pi$. Therefore the Fourier sine series converges pointwise at $x = \pi$ and $x = -\pi$ to $\frac{1}{2}[-\pi + \pi] = 0$. At $x \in (-\pi, \pi)$, $f_{\text{ext}}(x)$ is continuous (and equal to x) so the Fourier series converges pointwise to x.

c. The odd-periodic extension f_{ext} of f is discontinuous at $x = (2k+1)\pi$, $k \in \mathbb{Z}$ so the only point in $[0, \pi]$ where f_{ext} is discontinuous is $x = \pi$. The jump of f_{ext} at $x = \pi$ is $f_{\text{odd}}(\pi^+) - f_{\text{odd}}(\pi^-) = -2\pi$. Therefore, for $x \in [0, \pi]$ the Gibbs phenomenon appears at $x = \pi$ and the overshoot is approximately $0.09 \cdot (2\pi) \simeq 0.56$.

d. We have

$$A_n = \frac{2}{\pi} \int_0^\pi x \sin(nx)\, dx = \frac{2}{\pi} \int_0^\pi x \left(-\frac{1}{n}\cos(nx)\right)'\, dx.$$

Integration by parts gives

$$A_n = -\frac{2}{n\pi}\Big[x\cos(nx)\Big]_0^\pi + \frac{2}{n\pi}\int_0^\pi \cos(nx)\, dx = -\frac{2}{n\pi}\left[x\cos(nx) - \frac{\sin(nx)}{n}\right]_0^\pi.$$

Then

$$A_n = -(-1)^n \frac{2}{n}.$$

Question 6 (15 points)

Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1,\ 0 < t < \infty\}$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$.

a. (5 pt) Show that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.

b. (4 pt) Define the energy function $E(t) = \int_0^1 u(x,t)^2\, dx$ and show that $E(t)$ is a decreasing function of t, that is, $dE/dt \le 0$.

c. (6 pt) Show that $u(x,t) = u(1-x,t)$ for all $t \ge 0$ and $0 \le x \le 1$.
[Hint: define $w(x,t) = u(x,t) - u(1-x,t)$]

Solution

a. The strong maximum principle (as stated in the book) says that the solution u of the diffusion equation in a rectangle $R = [0,1] \times [0,T]$ where $T > 0$ attains its maximum only at the part of the boundary given by $B = \{x = 0 \text{ or } x = 1 \text{ or } t = 0\}$ (unless the solution is constant). Given that the solution is not constant (since $u(x,0) = 4x(1-x)$) we conclude that for any $(x,t) \in \text{int}(R)$ we have

$$\min_{(x,t)\in B} u(x,t) < u(x,t) < \max_{(x,t)\in B} u(x,t).$$

But for any $T > 0$ we have $\max_{(x,t)\in B} u(x,t) = 1$ and $\min_{(x,t)\in B} u(x,t) = 0$. Therefore,

$$0 < u(x,t) < 1$$

for any $0 < x < 1$ and $t > 0$.

b. We compute

$$\frac{dE}{dt} = \int_0^1 \frac{\partial}{\partial t}(u^2)\, dx = 2\int_0^1 u u_t\, dx.$$

Using the diffusion equation we rewrite the last expression as

$$\frac{dE}{dt} = 2\int_0^1 u u_{xx}\, dx,$$

and using integration by parts we get

$$\frac{dE}{dt} = 2[(uu_x)|_{x=1} - (uu_x)|_{x=0}] - 2\int_0^1 (u_x)^2\, dx.$$

Since $u(0,t) = u(1,t) = 0$ we have

$$\frac{dE}{dt} = -2\int_0^1 (u_x)^2\, dx \le 0.$$

c. Define $w(x,t) = u(x,t) - u(1-x,t)$. Then

$$w_t(x,t) = u_t(x,t) - u(1-x,t),$$

$$w_x(x,t) = u_x(x,t) + u_x(1-x,t),$$

and

$$w_{xx}(x,t) = u_{xx}(x,t) - u_{xx}(1-x,t) = u_t(x,t) - u_t(1-x,t),$$

where we used that u is a solution of the diffusion equation $u_t = u_{xx}$. Therefore

$$w_t = w_{xx}.$$

Furthermore,

$$w(0,t) = u(0,t) - u(1,t) = 0, \quad w(1,t) = u(1,t) - u(0,t) = 0,$$

and

$$w(x,0) = u(x,0) - u(1-x,0) = 4x(1-x) - 4(1-x)(1-(1-x)) = 0.$$

Therefore, w satisfies the diffusion equation while it is zero at the boundary. This means that $w = 0$ for all $0 \leq x \leq 1$ and $t \geq 0$. To be more explicit one can say that $w = 0$ is an obvious solution and then use uniqueness, or use the maximum principle to obtain that the maximum of w in the domain is 0 and the minimum is also 0 so $w = 0$ everywhere in the domain. This means that $u(x,t) = u(1-x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.